UDC 517.9

ON THE ACCURACY OF A NONLINEAR CLOSED SERVOMECHANISM

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We investigate the question of the accuracy of a closed servomechanism with a plant described by a linear second-order differential equation and a preselected control law with a "saturation"-type nonlinearity. We derive an algorithm for determining the exact value of the maximal cumulative error on an infinite time interval.

1. We consider a controlled system whose output $y(t)$ has to reproduce hitherto unknown input signals $x(t)$ from a class X of functions with bounded variation rate $|x^{\cdot}(t)| \leqslant m_0$, $x(0) = 0$. The behavior of the controlled plant under the action of the control signal v is described by the equation

$$Ty^{\cdot\cdot} + y^{\cdot} = v, \qquad y(0) = y^{\cdot}(0) = 0 \tag{1.1}$$

As the control law we take the hard feedback

$$v(\varepsilon) = k\varepsilon \quad (|k\varepsilon| \leqslant u_0), \qquad v(\varepsilon) = u_0 \operatorname{sign} \varepsilon \quad (|k\varepsilon| > u_0) \tag{1.2}$$

$$\varepsilon(t) = x(t) - y(t) \qquad (k > 0)$$

where u_0 is the constraint on the control signal. Subsequently we assume that $m_0 < u_0$. The system's block diagram is shown in Fig. 1.

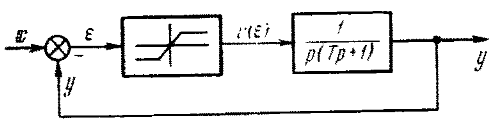

Fig. 1.

An upper bound for the quantity ε_∞, where

$$\varepsilon_\infty = \lim_{t\to\infty} \{\max_{x\in X} \max_{\tau\in[0,t]} |\varepsilon(\tau)|\} \tag{1.3}$$

was derived in [1] for the control law (1.2) and for a controlled plant described by an n th-order equation with time lag. Below, for a plant described by a second-order equation we derive an algorithm for finding the exact value of the quantity ε_∞, where we make use of another definition (which follows from stationarity), equivalent to (1.3),

$$\varepsilon_\infty = \max_{x\in X} \{\max_{t\in[0,\infty]} |\varepsilon(t)|\}$$

2. We denote $y^{\cdot} = z$, $x^{\cdot} = u$. We call u the control, and then the control u by which ε_∞ is realized is the optimal control. We write the equations of the closed system as

$$\varepsilon^{\cdot} = -z + u, \qquad z^{\cdot} = -\frac{1}{T} z + \frac{1}{T} v(\varepsilon), \qquad |u| \leqslant m_0, \qquad \varepsilon(0) = z(0) = 0 \qquad (2.1)$$

and we consider them on a phase plane (Fig. 2).

Theorem. The control u^* which takes the value $-m_0$ in the region $z > k\varepsilon$ and the value $+m_0$ in the region $z < k\varepsilon$ is optimal (the control u^* is not defined on the line $z = k\varepsilon$ itself).

Proof. Let u° be an arbitrary control satisfying $|u^\circ| \leqslant m_0$. At the origin we set u^* equal to $m_0 \operatorname{sign} u^\circ(t_1)$, where t_1 is the first instant at which $u^\circ \neq 0$; then for either of the controls u^* and u° the representative point, starting at the origin, finds itself in one and the same region (either $z < k\varepsilon$ or $z > k\varepsilon$), intersects the line $z = k\varepsilon$, continues the motion in the other region, etc. We denote by A_i^* and A_i°, respectively, the points of the ith intersection of the corresponding phase trajectories with the line $z = k\varepsilon$ and their coordinates by z_i^* and z_i° (the points A_0^* and A_0° coincide with the origin). We assert that the inequalities $|z_i^\circ| \leqslant |z_i^*| \leqslant u_0$ are valid for all points A_i^* and A_i°. We prove this by induction.

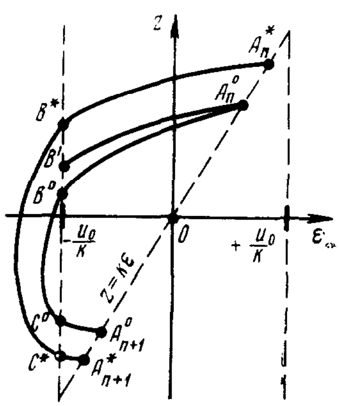

Fig. 2.

The inequalities are valid for $i = 0$ ($z_0^* = z_0^\circ = 0$). Let us assume that the inequalities are valid for $i = n$

$$|z_n^\circ| \leqslant |z_n^*| \leqslant u_0 \qquad (2.2)$$

and prove them for $i = n + 1$. Consider the phase trajectory portions $A_n^* A_{n+1}^*$ and $A_n^\circ A_{n+1}^\circ$ (Fig. 2). For the sake of definiteness we assume that they lie in the region $z > k\varepsilon$. Each of these portions can either intersect the line $\varepsilon = -u_0/k$ or not intersect it. Thus, four combinations of these two cases are possible. Let us consider only one of them, namely: both portions intersect the line $\varepsilon = -u_0/k$ (the proof is analogous for the other combinations). We denote by B^*, C^* the points of the first and last intersections of the line $\varepsilon = -u_0/k$ with $A_n^* A_{n+1}^*$, and by B°, C° with $A_n^\circ A_{n+1}^\circ$. The points B^*, C^*, B°, C° divide the corresponding phase trajectory portions into three segments. Consider the first segments: $A_n^* B^*$ and $A_n^\circ B^\circ$. We denote by B' the point of first intersection of the line $\varepsilon = -u_0/k$ with the phase trajectory starting from A_n° with $u = -m_0$. Since $z > k\varepsilon$, from (2.1) follows

$$d\varepsilon / dz|_{u=-m_0} \geqslant d\varepsilon / dz|_{u=u^\circ}$$

Hence $z_{B'} \geqslant z_{B^\circ}$ (where $z_{B'}$, z_{B° are the ordinates of points B', B°). Taking into account that $A_n^* B^*$ and $A_n^\circ B'$ cannot intersect (except in the case of identical coincidence) we obtain $z_{B^*} \geqslant z_{B'}$ and, hence $z_{B^*} \geqslant z_{B^\circ}$. Furthermore, from Eqs. (2.1) and inequalities (2.2) follow $-m_0 \leqslant z_{B^*} \leqslant u_0$, $-m_0 \leqslant z_{B^\circ} \leqslant u_0$. We now consider the next segments: $B^* C^*$ and $B^\circ C^\circ$. Proceeding from the inequalities just obtained and reasoning analogously, we can show that $z_{C^*} \leqslant z_{C^\circ}$, $-u_0 \leqslant z_{C^*} \leqslant -m_0$, $-u_0 \leqslant z_{C^\circ} \leqslant m_0$. In exactly the same way we can show that

$$|z_{n+1}^\circ| \leqslant |z_{n+1}^*| \leqslant u_0 \qquad (2.3)$$

for the last segments $C^* A_{n+1}^*$ and $C^\circ A_{n+1}^\circ$. The assertion is proved.

We denote by ε_i^* and ε_i° the maximum of $|\varepsilon|$ as we move from the points A_i^* and A_i° upto the first intersection with the line $z = k\varepsilon$ under controls u^* and u°, respectively. Above we derived (2.3) from (2.2). By the same arguments, from (2.2) we can derive the inequality

$$\varepsilon_n^\circ \leqslant \varepsilon_n^* \qquad (2.4)$$

Thus we have two sequences: $\{\varepsilon_n^\circ\}$ and $\{\varepsilon_n^*\}$, and from the way they were constructed there follows ($t \in [0, \infty]$) :

$$\sup_n \varepsilon_n^\circ = \max |\varepsilon(t)| \quad (u = u^\circ), \qquad \sup_n \varepsilon_n^* = \max |\varepsilon(t)| \quad (u = u^*) \qquad (2.5)$$

From (2.4) ensues $\sup_n \varepsilon_n^\circ \leqslant \sup_n \varepsilon_n^*$. Because u° is arbitrary, the theorem's validity follows from this and from (2.5).

3. For $u = u^*$ a limit cycle is possible in the system described by Eqs. (2.1) and, moreover, the absolute value of the maximal deviation along the coordinate ε in this limit cycle it is the desired quantity ε_∞. For finding the limit cycle we can make use of the method of point transformations [2], where as a consequence of the symmetry about the origin it is sufficient to examine only one-half of this cycle: to be specific let us examine the left half. We denote the points of successive intersections of a certain portion of the phase trajectory for $u = u^*$ with the lines $z = k\varepsilon$ and $\varepsilon = -u_0 / k$ by B_1, B_2, B_3, B_4 (Fig. 3) and their ordinates by S_1, S_2, S_3, S_4. According to [2] the sequence of steps to be taken to determine the limit cycle are as follows:

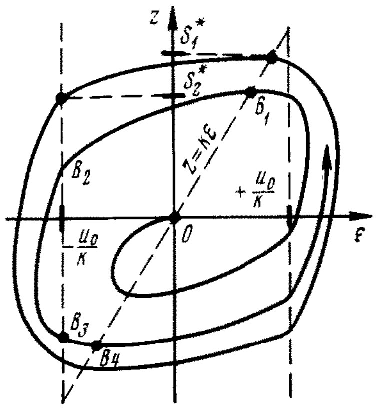

Fig. 3.

1. Find the functions $S_1(\tau_1)$, $S_2(\tau_1)$, $S_2(\tau_2)$, $S_3(\tau_2)$, $S_3(\tau_3)$, $S_4(\tau_3)$, where τ_1 is the time taken to move from B_1 to B_2, τ_2 from B_2 to B_3, τ_3 from B_3 to B_4. It is not difficult to obtain these functions because Eq. (2.1) is linear on the corresponding segments. We present only the final results:

$$S_1(\tau_1) = -m_0 - \frac{u_0 - m_1}{k g_1(\tau_1)} e^{\mu\tau_1}, \qquad g_1(\tau_1) = \frac{1}{k}\cos\nu\tau_1 + \frac{\mu/k - 1}{\nu}\sin\nu\tau_1 \qquad (3.1)$$

$$S_2(\tau_1) = -m_1 - \frac{u_0 - m_0}{k g_1(\tau_1)}\left[\cos\nu\tau_1 + \frac{(\mu^2+\nu^2)/k - \mu}{\nu}\sin\nu\tau_1\right]$$

$$S_2(\tau_2) = (u_0 - m_0)\tau_2[T(1 - e^{-\tau_2/T})]^{-1} - u_0, \qquad S_3(\tau_2) = (u_0 - m_0)\tau_2[T(e^{\tau_2/T} - 1)]^{-1} - u_0 \qquad (3.2)$$

$$S_3(\tau_3) = -m_1 + \frac{u_0 - m_0}{k g_2(\tau_3)}\left[-\cos\nu\tau_3 + \frac{\mu^2+\nu^2-\mu}{k\nu}\sin\nu\tau_3\right]$$

$$S_4(\tau_3) = -m_0 + \frac{u_0 - m_1}{g_2(\tau_3)} e^{-\mu\tau_3}\left[-\frac{1}{k}\cos^2\nu\tau_3 + \frac{1}{\nu^2}\left(\frac{\mu^2-\nu^2}{k} + \mu\right)\sin^2\nu\tau_3\right] \qquad (3.3)$$

$$g_2(\tau_3) = \frac{\cos\nu\tau_3}{k} + \frac{1}{\nu}\left(1 - \frac{\mu}{k}\right)\sin\nu\tau_3$$

where $-\mu \pm i\nu$ are the complex conjugate roots of the equation $T\lambda^2 + \lambda + k = 0$. The case of real roots is analyzed below.

2. Taking τ_1, τ_2 as parameters, from (3.1), (3.2) construct the graphs $S_1 = S_1(S_2)$, $S_3 = S_3(S_2)$ and on their basis, taking S_2 as a parameter, construct the graph $-S_1 = -S_1(S_3)$.

3. Taking τ_3 as a parameter, from (3.3) construct the graph $S_4 = S_4(S_3)$ and find the point of intersection of this graph with the graph $-S_1 = -S_1(S_3)$. The ordinate of this point of intersection yields the desired parameter of the limit cycle: $S_1{}^*$. Using this determine the parameter $S_2{}^*$ from the graph $S_1 = S_1(S_2)$. Further, we can express ε_∞ in terms of $S_2{}^*$

$$\varepsilon_\infty = \frac{u_0}{k} + T(m_0 + S_2{}^*) - T(u_0 - m_0)\ln\frac{S_2{}^* + u_0}{u_0 - m_0} \qquad (3.4)$$

We note that the estimate of the quantity ε_∞ presented in [1] for a plant described by Eq. (1.2) would have the form

$$\varepsilon_\infty \leqslant \frac{u_0}{k} + T(m_0 + u_0) - T(u_0 - m_0)\ln\frac{2u_0}{u_0 - m_0}$$

It differs from (3.4) in that $S_2{}^*$, the ordinate of the point of intersection of the limit cycle with the line $\varepsilon = -u_0 / k$, is replaced by its upper bound, the value u_0.

We present the results of a numerical calculation. For $m_0 = 1$, $u_0 = 2$, $T = 3$ and as k takes the successive values 1, 2, 3, 4, ∞, the exact values of the maximal cumulative error ε_∞ are: 5.30, 4.85, 4.71, 4.65, 4.54, while the estimates presented in [1] yield, respectively: 6,84, 5,84, 5,50, 5,34, 4.84. Calculation results for other values of the parameters give values of ε_∞ lying between the limits from 50 to 95% of the estimates presented in [1].

4. Let us consider a system in which the control law is linear: $v(\varepsilon) = k\varepsilon$. For such a system it has been shown [3] that

$$\varepsilon_\infty = \frac{m_0}{k} \qquad \left(k < \frac{1}{4T}\right) \qquad (4.1)$$

$$\varepsilon_\infty = \zeta(m_0, k, T), \qquad \zeta = \frac{m_0}{k} + 2\sqrt{\frac{T}{k}}\, m_0 \exp\left(-\frac{\pi - \psi}{2T\nu}\right)\left[1 - \exp\left(-\frac{\pi}{2T\nu}\right)\right]^{-1}$$

$$\psi = \operatorname{arctg}(2T\nu) \qquad \left(k \geqslant \frac{1}{4T}\right)$$

We analyze the case $k < 1/(4T)$. Here $|k\varepsilon| \leqslant m_0 < u_0$, i.e., the addition of the nonlinear constraint (1.2) does not affect the system's operation. Hence $\varepsilon_\infty = m_0 / k$ in (4.1) is valid for the original nonlinear system described by (1.1)(1.2) when $k < 1/(4T)$.

We analyze the case $k \geqslant 1/(4T)$. Here for $\zeta \leqslant u_0 / k$ as before, $|k\varepsilon| \leqslant u_0$, and the ε_∞ of the original nonlinear system can be computed from formula (4.1) for the linear system, whence we see that ε_∞ decreases monotonically as k grows. For $\zeta > u_0 / k$ the system goes onto the saturation segment of characteristics (1.2), and the ε_∞ can be computed by means of the algorithm indicated above. Examples computed for different values of k permit us to expect here that ε_∞ decreases monotonically as k grows, although we have not succeeded in proving this.

BIBLIOGRAPHY

1. Gnoenskii, L. S., The accuracy of certain nonlinear control systems with restrictions and lag. PMM Vol. 34, №6, 1970.
2. Andronov, A. A., Vitt A. A. and Khaikin S. E., Theory of Oscillations. Moscow, Fizmatgiz, 1959.
3. Gnoenskii, L. S., Kamenskii G. A. and El'sgol'ts L. E., Mathematical Foundations of the Theory of Controlled Systems. Moscow, "Nauka", 1969.

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